Riemannian Geometry | Week 7

Let (M, g) be a Riemannian manifold.

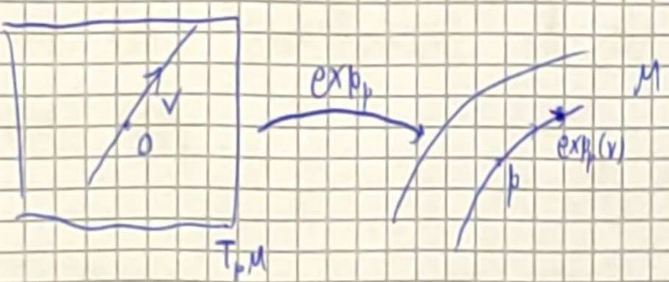
$\forall p \in M$, $v \in T_pM$: $\gamma_{p,v}(t)$ is the maximal geodesic

with $\gamma(0) = p$, $\dot{\gamma}(0) = v$.

Last time: We defined the exponential map

$$\exp_p : \Omega_p \subseteq T_pM \longrightarrow M$$

$$\exp_p(tv) = \gamma_{p,v}(t)$$



We also proved that in a small neighborhood of $v = 0$:

$\exp_p : U \longrightarrow \exp_p(U) \subseteq M$ is a diffeomorphism on its image,

since $\boxed{d\exp_p\big|_{v=0} = id_{T_pM}}$

- Injectivity radius:

$$i(p) = \sup\{r > 0 : \exp_p \text{ is a diffeo on its image when restricted to } B_r(0) = \{v : \|v\|_g < r\}\}$$

- The map $F : (p, v) \in TM \longrightarrow (p, \exp_p(v)) \in M \times M$ defines a diffeomorphism from a neighborhood of the 0 section of TM to a neighborhood of the diagonal $\Delta \subset M \times M$.

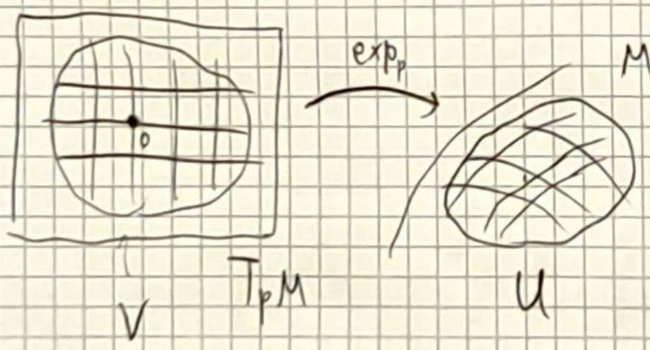
Normal coordinates

The coordinate system in which g looks as flat as possible.

Fix an orthonormal (with respect to g) basis $e_1, \dots, e_n$ of T_pM.
This fixes Cartesian coordinates on $T_pM \cong \mathbb{R}^n$:

$$\forall\, w \in T_pM, \quad w = \underbrace{w^i}_{\text{Coordinates}} e_i$$

Normal coordinates in a neighborhood of p in M are the "push-forward" of the Cartesian coordinates on T_pM via $\exp_p$:



If U is a neighborhood of p in M such that $\exp_p^{-1}$ is defined there:

$\forall\, q \in U$, if $q = \exp_p(v)$

$$\boxed{x^i(q) = v^i = \left(\exp_p^{-1}(q)\right)^i}$$

$$\Leftrightarrow \quad x^i(\exp_p(v)) = v^i$$

In these coordinates: $\exp_p : V \to U$ is the identity

And: $x^i(\exp_p(tv)) = t \cdot v^i \;\Rightarrow\; \gamma^i_{p,v}(t) = t \cdot v^i$

(In these coordinates: Geodesics through p are straight lines)

Proposition

In normal coordinates centered at p,

$$g_{ij}(0) = \delta_{ij}, \quad \partial_i g_{jk}(0) = 0, \quad \Gamma^k_{ij}(0) = 0$$

(But in general: $\partial^2 g(0) \neq 0 \Rightarrow$ this is related to the Riemann curvature tensor).

Proof: Since $x^i(\exp_p(v)) = v^i \Rightarrow x^i(\exp_p(t \cdot e_j)) = t \cdot \delta^i_j$

So $\forall j = 1, \dots, n$, the curve

$t \longrightarrow \exp_p(t \cdot e_j)$ is the x^j-coordinate curve

$\Rightarrow \frac{\partial}{\partial x^j}$ is the tangent of this curve.

So $\frac{\partial}{\partial x_j}\Big|_p = \frac{d}{dt}(\exp_p(te_j))\Big|_{t=0} = d\exp_p\Big|_{v=0}(e_j) = e_j$

So $g_{ij}(0) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\Big|_p = g(e_i, e_j) = \delta_{ij}$

since $\{e_i\}_{i=1}^n$ is an orthonormal basis.

The curve $\gamma_{p,v}(t) = \exp_p(tv)$ is a geodesic $\forall v$.

$\Rightarrow$ In normal coordinates: $\gamma^k_{p,v}(t) = t \cdot v^k$

So $\ddot{\gamma}^k_{p,v}(t) + \Gamma^k_{ij}(\gamma_{p,v}(t))\, \dot{\gamma}^i_{p,v} \cdot \dot{\gamma}^j_{p,v} = 0$

$\Rightarrow \Gamma^k_{ij}(\gamma_{p,v}(t)) \cdot v^i v^j = 0 \overset{t=0}{\Longrightarrow} \Gamma^k_{ij}(0)\, v^i v^j = 0 \quad \forall\, v \in T_pM$

Since $\Gamma^k_{ij}(0)$ is symmetric in $i, j \rightarrow \Gamma^k_{ij}(0) = 0$

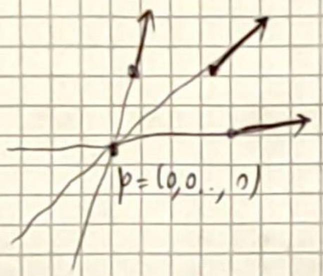
And: $\partial_k g_{ij} = \partial_k \langle \partial_i, \partial_j \rangle = g_{aj} \Gamma^a_{ki} + g_{ai} \Gamma^a_{kj}$

So $\partial_k g_{ij}(0) = 0$. $\boxed{}$

But we got more information from the proof:

$$\Gamma^k_{ij}(t \cdot v) \cdot v^i v^j = 0$$

So



The "radial" directions are special!!

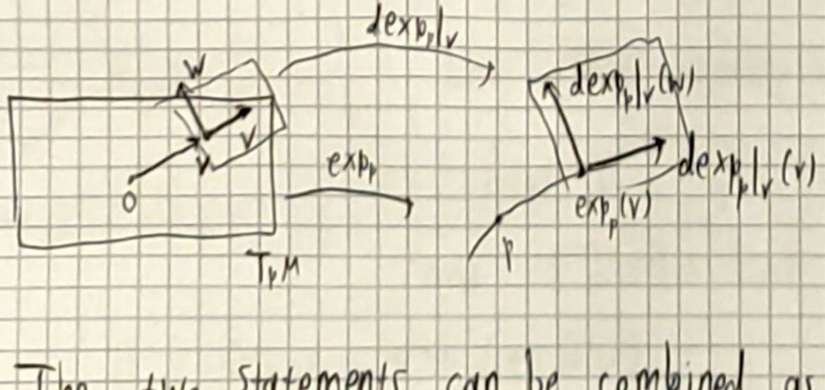
The Gauss lemma:

<u>Theorem</u>: The exponential map satisfies the following properties:

a) $\| \underbrace{d\exp_p|_v (v)}_{\in T_{\exp_p(v)} M} \| = \| \underset{\in T_pM}{v} \|$

b) $\forall \, w \in T_v(T_pM)$ orthogonal to v, we have:

$$\langle d\exp_p|_v (v), d\exp_p|_v (w) \rangle = 0$$



- The two statements can be combined as follows:

$\forall \, v \in \Omega_p$, $\forall w \in T_v \Omega_p$: $\langle d\exp_p|_v (v), d\exp_p|_v (w) \rangle = \langle v, w \rangle$

Remark: In normal coordinates,

$$S = d\exp_p\big|_v (v) \text{ is just the radial vector field}$$

(At the point $(x^1, \dots, x^n)$, $S^i = x^i$)

So the above says that $g_{ij}(x)\, x^i w^j = \delta_{ij}\, x^i w^j \quad \forall w$

$$\Rightarrow \boxed{g_{ij}(x)\, x^i = \delta_{ij}\, x^i}$$

Proof:

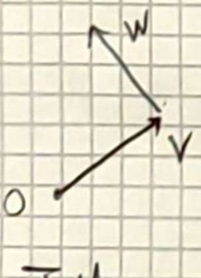
a) Recall: $\exp_p(tv) = \gamma_{p,v}(t)$

So $\dot{\gamma}_{p,v}(t) = \frac{d}{dt}\left(\exp_p(tv)\right) = d\exp_p\big|_{tv}(\dot{tv}) = d\exp_p\big|_{tv}(v)$

For $t = 1$: $\dot{\gamma}_{p,v}(1) = d\exp_p\big|_v (v)$

But $\gamma_{p,v}$ is a geodesic, so $\|\dot{\gamma}_{p,v}(1)\| = \|\dot{\gamma}_{p,v}(0)\| = \|v\|$.

b) We will use the first variation formula for the length:



T_pM

Without loss of generality $\|w\| = \|v\|$.

Let $\lambda(s) = \cos(s) \cdot v + \sin(s) \cdot w$

$\lambda(0) = v, \quad \dot{\lambda}(0) = w$

and $\|\lambda(s)\| = \|v\|$ (since $v \perp w$, $\|v\| = \|w\|$)

If $\phi(t,s) = \exp_p(t \cdot \lambda(s))$:

Variation of the curve $\gamma_{p,v}(t)$

and $\forall s$: $\phi_s(t)$ is a geodesic (acceleration = 0)

By 1^{st} variation formula of the length:

$$\frac{d}{ds} \ell(\phi_s)\Big|_{s=0} = \frac{1}{\ell(\gamma_{p,v})} \cdot \left\langle \frac{\partial \phi}{\partial s}\Big|_{s=0,\, t=1}, \dot{\gamma}(1) \right\rangle \quad ①$$

But: $\ell(\phi_s) = \int_0^1 \left\| \frac{\partial \phi_s}{\partial t} \right\| dt = \int_0^1 \|v\| \, dt = \|v\|$

And $\dot{\gamma}(1) = d\exp_p\big|_v (v)$

$$\frac{\partial \phi}{\partial s}\Big|_{t=1} = \frac{\partial}{\partial s}\left(\exp_p(t \cdot \lambda(s))\right)\Big|_{t=1} = d\exp_p\big|_{\lambda(s)} (\dot{\lambda}(s))$$

$$\Rightarrow \frac{\partial \phi}{\partial s}\Big|_{s=0,\, t=1} = d\exp_p\big|_v (w)$$

So $① \Rightarrow 0 = \left\langle d\exp_p\big|_v (v), d\exp_p\big|_v (w) \right\rangle$ $\blacksquare$

In particular:

The exponential map send spheres centered at 0 in T_pM to surfaces orthogonal to the geodesics $\gamma_{p,v}$

- We consider the normal coordinates on neighborhoods U of p which are <u>normal</u> (any $q \in U$ can be connected with a geodesic segment to p). Intersection of such neighborhoods: ~~Totally geod~~ <u>Convex neighborhood</u>.

Polar coordinates:

$e_2\| = 1$

Instead of using Cartesian coordinates on T_pM, we could use polar coordinates and push them forward on M via $\exp_p$.

Assume $\dim M = 2$, and e_1, e_2 orth. frame at p.

Let $W_p = \exp_p\left(\{v \in T_pM : \|v\| < i(p)\}\right)$

polar coordinates (r,θ) on $W_p \setminus \{p\}$:

$$\left.\begin{array}{l} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{array}\right\} \quad \begin{array}{l} r = \sqrt{x^2+y^2} \\ \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}} \end{array}$$

(x,y): Normal coordinates

Proposition: In Polar coordinates:

$$g = dr^2 + b^2(r,\theta)\,d\theta^2,$$

$\Leftarrow g_{rr} = 1,\ g_{r\theta} = 0$

and $\lim_{r\to 0} b(r,\theta) = 0$, $\lim_{r\to 0} \frac{\partial b}{\partial r}(r,\theta) = 1$.

Rmk: On the flat plane: $g_E = dr^2 + r^2 d\theta^2$

Proof: In polar coordinates: $q(r,\theta) = \exp_p(r\cdot\cos\theta\cdot e_1 + r\cdot\sin\theta\cdot e_2)$

$= \exp_p(r\cdot(\cos\theta\, e_1 + \sin\theta\, e_2))$

So $r \longrightarrow q(r,\theta)$ is a geodesic

$\Rightarrow$ Geodesic tangent vector: $\frac{\partial}{\partial r}$

So $\left\|\frac{\partial}{\partial r}\right\| = \text{const}$. As $r \to 0$: $\left\|\frac{\partial}{\partial r}\right\| = \|\cos\theta \cdot e_1 + \sin\theta \cdot e_2\| = 1$.

So $g_{rr} = 1$ everywhere.

and:

$$\left.\frac{\partial}{\partial r}\right|_{\exp_p(v)} = d\exp_p\big|_v \left(\cos\theta \cdot e_1 + \sin\theta \, e_2\right)$$

$$\left.\frac{\partial}{\partial \theta}\right|_{\exp_p(v)} = d\exp_p\big|_v \left(r \cdot (-\sin\theta \cdot e_1 + \cos\theta \cdot e_2)\right)$$

By Gauss lemma: $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = 0$.

So: $g = dr^2 + b^2 \, d\theta^2$, $b = \left\|\frac{\partial}{\partial \theta}\right\|$

By change of coordinates:

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r \cdot \cos\theta \frac{\partial}{\partial y}$$

So $\frac{b(r,\theta)}{r} = \sqrt{g_{xx}\sin^2\theta + 2g_{xy}\cos\theta \cdot \sin\theta + g_{yy}\cos^2\theta}$

At $r = 0$: $g_{ij} = \delta_{ij}$,

So $\frac{b(r,\theta)}{r} \longrightarrow 1 \quad \forall \theta$ $\square$